Problem 2

A student forgot the Product Rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, he was lucky and got the correct answer. The function f that he used was $f(x) = e^{x^2}$ and the domain of his problem was the interval $\left(\frac{1}{2}, \infty\right)$. What was the function g?

Solution

The Product Rule says that

$$(fg)' = f'g + g'f.$$

Since the student got the correct answer, it must be the case that

$$f'g + g'f = f'g', \tag{1}$$

and because we know f (and consequently f'), this will be a differential equation for g that we can solve.

$$f(x) = e^{x^2} \quad \rightarrow \quad f'(x) = e^{x^2} \cdot 2x$$

Plugging these functions into (1) yields the following:

$$\begin{aligned} 2x\cancel{e^{x^2}}g + g'\cancel{e^{x^2}} &= 2x\cancel{e^{x^2}}g' \\ 2xg &= (2x-1)g' \\ (2x-1)g' &= 2xg \end{aligned}$$

Separate variables.

$$\begin{aligned} (2x-1)\frac{dg}{dx} &= 2xg \\ \frac{dg}{g} &= \frac{2x}{2x-1}\,dx \end{aligned}$$

Integrate both sides.

$$\ln|g| = \int \frac{2x}{2x-1}\,dx$$

To solve the integral on the right, make the u-subtitution

$$\begin{aligned} u = 2x-1 \quad &\rightarrow \quad u+1 = 2x \\ du = 2\,dx \quad &\rightarrow \quad \frac{1}{2}du = dx. \end{aligned}$$

And so

$$\begin{aligned} \ln|g| &= \frac{1}{2}\int \frac{(u+1)}{u}\,du \\ \ln|g| &= \frac{1}{2}\int \left(1 + \frac{1}{u}\right)du \\ \ln|g| &= \frac{1}{2}\left(u + \ln|u|\right) + C \end{aligned}$$

Exponentiate both sides.

$$|g(x)| = e^{\frac{1}{2}(2x-1+\ln|2x-1|)}e^{C}$$

Because the domain is $\left(\frac{1}{2}, \infty\right)$, the absolute value signs around $2x - 1$ can be dropped.

$$g(x) = \pm e^{C} e^{x-\frac{1}{2}+\ln(2x-1)^{1/2}}$$
$$g(x) = Ae^{x-\frac{1}{2}}\sqrt{2x-1}$$

Therefore, the function the student worked with is

$$g(x) = Ae^{x-\frac{1}{2}}\sqrt{2x-1}.$$